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ON THE APPROXIMATE SOLUTION OF THE THIRD BOUNDARY-VALUE PROBLEM OF HEAT-CONDUCTION THEORY FOR A CIRCLE

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An efficient analytical approximate representation of the solution of the third boundary-value problem of heat-conduction theory for a circle is obtained. A uniform evaluation of the error of the approximate formula ensures the convergence of the numerical algorithm.

Keywords: *heat conduction, heat exchange, boundary-value problem, integral equation, approximate formula.*

Introduction. Boundary-value problems in the theory of stationary heat conduction are known to be one of the most important classes of the problems of mathematical physics. The third boundary-value problem is the most general and important. Boundary conditions of the third kind (heat exchange with the ambient medium by the Newton law) are widely used in analytical investigations of heat conduction in solid bodies in liquid or gas flow and in thermoelasticity, when the influence of the temperature on the stressed state of deformable solid bodies is investigated.

We consider the third basic boundary-value problem of heat-conduction theory for the Laplace equation in a unit circle with the center at the origin of coordinates

$$\Delta T = 0 \,, \quad r < 1 \,, \tag{1}$$

$$\lambda \left.\frac{\partial T}{\partial r}\right|_{r=1} = \alpha a(\varphi) \left[T(r, \varphi)\big|_{r=1} - b(\varphi) T_{\mathrm{m}} \right] , \tag{2}$$

where $a(\varphi)$ and $b(\varphi)$ are the functions prescribed on the segment $[-\pi, \pi]$.

The approximate representation of the complex thermal potential of the boundary-value problem (1) and (2), which enables one to determine the basic elements of the heat flux, has been obtained in [1] using the methods of analytical-function theory and special formulas for the Schwartz and Hilbert integrals.

At the same time, the physical quantity characterizing the process of heat conduction is temperature. Below, we construct the approximate representation of the temperature function by polylogarithms separately for it using the Dini integral. The approximate formula obtained in the work is comparatively simple and stable, whereas the uniform evaluations of errors suggest the convergence of the computational process.

Formulation of the Problem. The work seeks to obtain the effective approximate representation of the solution of the third boundary-value problem of heat-conduction theory for a circle with a variable heat-transfer coefficient.

Integral Equation of the Boundary-Value Problem (1) and (2). We write boundary condition (2) in the form

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=1} + q(\varphi)\,\theta = f(\varphi) \,, \tag{3}$$

where

$$\theta = \frac{T}{T_0} \,; \quad q(\varphi) = -\,\mathrm{Bi}\, a(\varphi) \,; \quad f(\varphi) = \frac{T_{\mathrm{m}}}{T_0}\, q(\varphi)\, b(\varphi) \,; \quad \mathrm{Bi} = \left.\frac{\alpha r}{\lambda}\right|_{r=1} .$$

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It is well known (see, e.g., [2, p. 280; 3, p. 319] that if we have $q(\varphi) > 0$ on the interval $[-\pi, \pi]$ in boundary condition (3), this boundary-value problem has a unique solution. We will assume that this condition is observed.

We use the representation of the solution of the Neumann problem for a circle (see, e.g., [4, pp. 598–600]) by the Dini integral. Then, inside the circle, we have

$$\theta = \theta(r, \varphi) = -\frac{1}{\pi}\int_{-\pi}^{\pi} q(\tau)\,\theta(\tau)\ln|t - z|\,d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\ln|t - z|\,d\tau + C, \tag{4}$$

where $t = \exp(i\tau)$, $z = r\exp(i\varphi)$, and C is a constant equal to the value of the function θ at the center of the circle.

Passing to the limit at z tending to the points of the circle in equality (4), we obtain the integral relation for the boundary values of the function $\theta = \theta(\varphi)$:

$$\theta(\varphi) = -\frac{1}{\pi}\int_{-\pi}^{\pi} q(\tau)\,\theta(\tau)\ln|t - \exp(i\varphi)|\,d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\ln|t - \exp(i\varphi)|\,d\tau + C. \tag{5}$$

Here, it is necessary that the integral of the boundary values of the normal derivative around the circle be equal to zero. Integrating the boundary-value equality (3), we find the condition

$$\int_{-\pi}^{\pi} q(\varphi)\,\theta(\varphi)\,d\varphi = \int_{-\pi}^{\pi} f(\varphi)\,d\varphi. \tag{6}$$

Noting that $|t - \exp(i\varphi)| = 2\sin\frac{\tau-\varphi}{2}$ and, according to the mean-value theorem,

$$C = \theta(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta(\varphi)\,d\varphi,$$

we obtain, from (5) with condition (6), the integral equation

$$\theta(\varphi) = -\frac{1}{\pi}\int_{-\pi}^{\pi} q(\tau)\,\theta(\tau)\ln\left|\sin\frac{\tau-\varphi}{2}\right|d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\ln\left|\sin\frac{\tau-\varphi}{2}\right|d\tau + \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta(\tau)\,d\tau.$$

We write this equation in the form

$$u(\varphi) + \frac{q(\varphi)}{\pi}\int_{-\pi}^{\pi} u(\tau)\ln\left|\sin\frac{\tau-\varphi}{2}\right|d\tau - \frac{q(\varphi)}{2\pi}\int_{-\pi}^{\pi} u(\tau)\frac{d\tau}{q(\tau)}\,d\tau = \frac{q(\varphi)}{\pi}\int_{-\pi}^{\pi} f(\tau)\ln\left|\sin\frac{\tau-\varphi}{2}\right|d\tau, \tag{7}$$

where the notation $u(\varphi) = q(\varphi)\,\theta(\varphi)$ has been introduced.

The kernel of the integral equation (7) has a logarithmic singularity. The Fredholm theory is applicable to equations with such singularities. Using the equality (see [1, p. 14])

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\ln\left|\sin\frac{\tau-\varphi}{2}\right|\right|d\tau = 2\ln 2$$

we establish the inequality

$$\left| \frac{q(\varphi)}{\pi} \int_{-\pi}^{\pi} u(\tau) \ln \left| \sin \frac{\tau - \varphi}{2} \right| d\tau - \frac{q(\varphi)}{2\pi} \int_{-\pi}^{\pi} u(\tau) \frac{d\tau}{q(\tau)} d\tau \right| \le (2 \ln 2 + I)\, q \max_{\varphi \in [-\pi,\pi]} |u(\varphi)| ,$$

where $I = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\tau}{q(\tau)}$ and $q = \max_{\varphi \in [-\pi,\pi]} q(\varphi)$.

Now we easily assure ourselves that if the functions $q(\varphi)$ and $f(\varphi)$ are continuous on the interval $[-\pi, \pi]$ and $(2 \ln 2 + I)q < 1$, the integral equation (7) will have a unique solution in the class of continuous functions.

Approximate Solution of the Integral Equation (7). One of the most efficient methods of approximate solution of integral equations is reducing it to a system of linear algebraic equations. Proof of the solvability of this system and of its stability is of importance for practical implementation of such a method.

We use a special quadratic formula for integrals with logarithmic kernels [1, pp. 56–61]. We replace the integrals in Eq. (7) by the corresponding quadrature sums with remainders. Thereafter we arrive at the equality

$$u(\varphi) - q(\varphi) \sum_{-n}^{n} A_k(\varphi)\, u(\varphi_k) - q(\varphi) \sum_{-n}^{n} C_k u(\varphi_k) = -\, q(\varphi) \sum_{-n}^{n} A_k(\varphi) f(\varphi) + q(\varphi) \left[E_u(\varphi) + E_c + E_f(\varphi)\right] , \tag{8}$$

where the coefficients are

$$A_k(\varphi) = -\frac{1}{\pi} \int_{\varphi_k - \frac{h}{2}}^{\varphi_k + \frac{h}{2}} \ln \left| \sin \frac{\tau - \varphi}{2} \right| d\tau ; \quad C_k = \frac{1}{2\pi} \int_{\varphi_k - \frac{h}{2}}^{\varphi_k + \frac{h}{2}} \frac{d\tau}{q(\tau)} ;$$

the nodes are $\varphi_k = kn$, $k = -n, ..., -1, 0, 1, ..., n$, $h = \frac{2\pi}{2n + 1}$, and $E_u(\varphi)$, E_c, and $E_f(\varphi)$ are the remainders of the corresponding quadratic formulas.

The coefficients $A_k(\varphi)$ have ben computed in [1, p. 60]:

$$A_k(\varphi) = \frac{1}{\pi} \left[h \ln 2 + N^2 \left(\varphi - \varphi_k + \frac{h}{2} \right) - N^2 \left(\varphi - \varphi_k - \frac{h}{2} \right) \right],$$

where $N^2(\varphi) = \sum_{k=1}^{\infty} \frac{\sin k\varphi}{k^2}$ are the values of the imaginary part of the Euler dilogarithm [5] on a unit circle. Also, we note that all the coefficients $A_k(\varphi)$, C_k ($k = -n, ..., -1, 0, 1, ..., n$, $\varphi \in [-\pi, \pi]$) are nonnegative and satisfy the relations

$$\sum_{-n}^{n} A_k(\varphi) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \ln \left| \sin \frac{\tau - \varphi}{2} \right| d\tau = 2 \ln 2 , \quad \sum_{-n}^{n} C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\tau}{q(\tau)} = I . \tag{9}$$

Satisfying equality (8) at the points φ_k, we obtain the system of equations

$$u_j - q(\varphi_j) \sum_{-n}^{n} A_k(\varphi_j)\, u_k - q(\varphi_j) \sum_{-n}^{n} C_k u_k = -\, q(\varphi_j) \sum_{-n}^{n} A_n(\varphi_j) f(\varphi_k) + q(\varphi_j) \left[E_u(\varphi_j) + E_c + E_c(\varphi_j)\right] , \tag{10}$$

$$j = -n, ..., -1, 0, 1, ..., n ,$$

where we have set $u_j = u(\varphi_j)$.

We discard the remainders on the right-hand side of (10); then, instead of system (10), we will have the algebraic linear system

$$u_j - q(\varphi_j) \sum_{-n}^{n} \left[A_k(\varphi_j) - C_k\right] \tilde{u}_k = -q(\varphi_j) \sum_{-n}^{n} A_k(\varphi_j) f(\varphi_k)\,, \tag{11}$$

$$j = -n, \ldots, -1, 0, 1, \ldots, n\,,$$

in which $\tilde{u}_k$ are the approximate values of $u(\varphi_k)$ involved in Eq. (10).

We will investigate system (11) by the methods of difference-scheme theory [6]. For the remainders of the quadrature formulas under study, we easily obtain the estimates

$$|E_u(\varphi)| \le 2 \ln 2\omega(u, h)\,, \quad |E_c| \le I\omega(u, h)\,, \quad |E_f(\varphi)| \le 2 \ln 2\omega(f, h)\,,$$

where $\omega(u, h)$ and $\omega(f, h)$ are the moduli of continuity of the functions $u(\varphi)$ and $f(\varphi)$ respectively.

Using these estimates we establish the inequality determining the approximation of the integral equation (7) by the system of linear algebraic equations (11):

$$|q(\varphi)| \left[E_u(\varphi) + E_c + E_f(\varphi)\right] \le q\left[(2 \ln 2 + I)\, \omega(u, h) + 2 \ln 2\omega(f, h)\right]. \tag{12}$$

Repeating the considerations given in [1, p. 103], with allowance for the relations for the coefficients of the quadrature formulas, we arrive at the following statement:

Theorem 1. *Let $q(\varphi)$ and $f(\varphi)$ be the functions continuous on the interval $[-\pi, \pi]$; we obtain*

$$(2 \ln 2 + I)\, q < 1\,,$$

then system (11) is uniquely solvable and stable and we have the estimate

$$\max_m |u_m(\varphi_m) - u_m| \le \frac{q\left[(2 \ln 2 + I)\, \omega(u, h) + 2 \ln 2\omega(f, h)\right]}{1 - (2 \ln 2 + I)\, q}\,. \tag{13}$$

Approximate Solution of the Boundary-Value Problem (1)–(2). We will construct it on the basis of formula (4), using the special approximate formula for the Dini integral [7]. We obtain from formula (4), allowing for the relationship of the functions $T(r, \varphi)$, $u(\varphi)$, and $\theta(\varphi)$ and for the necessary condition (6) for solvability of the boundary-value problem, the exact expression of the temperature distribution in the circle by the following relation:

$$T(r, \varphi) = T_0\left[\frac{1}{\pi}\int_{-\pi}^{\pi} u(\tau) \ln \frac{2}{|t - z|}\, d\tau - \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau) \ln \frac{2}{|t - z|}\, d\tau + \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\tau) \frac{d\tau}{q(\tau)}\right]. \tag{14}$$

Replacing the integrals on the right-hand side of formula (14) by the corresponding quadrature sums, we find the approximate representation of the solution of the boundary-value problem (1) and (2)

$$\tilde{T}(r, \varphi) = T_0\left[\sum_{-n}^{n} D_k(r, \varphi)\, \tilde{u}_k - \sum_{-n}^{n} D_k(r, \varphi) f(\varphi_k) + \sum_{-n}^{n} C_k \tilde{u}_k\right], \tag{15}$$

where we have the coefficients

$$D_k(r, \varphi) = \frac{1}{\pi}\left[h \ln 2 + \operatorname{Im}\left(L^2\left(z \exp\left[-i\left(\varphi_k - \frac{h}{2}\right)\right]\right) - L^2\left(z \exp\left[-i\left(\varphi_k + \frac{h}{2}\right)\right]\right)\right)\right];$$

$L^2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ (Euler dilogarithm [5]), $\tilde{u}_k$ are the solutions of system (11), and the coefficients C_k have been determined above.

N o t e. The series

$$L^s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

determine the polylogarithms of order s, i.e., the special functions studied in [8] and in [1] where the new formulas for approximate computation of Cauchy-type integrals and the solutions of certain boundary-value problems of mathematical physics have been obtained using polylogarithms. It is common practice to call the polylogarithm of second order ($s = 2$) the Euler dilogarithm.

Evaluation of the Error of the Approximate Formula (15). **Theorem 2.** *If the functions $a(\varphi)$ and $b(\varphi)$ in boundary condition (2) are continuous on the interval $[-\pi, \pi]$, we have the following, uniform in r and φ ($r \le 1$ and $-\pi \le \varphi \le \pi$), estimate of the error of the approximate formula (15)*:

$$\left| T(r, \varphi) - \tilde{T}(r, \varphi) \right| \le T_0 \frac{(2 \ln 2 + I)\, \omega(u, h) + 2 \ln 2\omega(f, h)}{1 - (2 \ln 2 + I)\, q} . \tag{16}$$

Proof. Comparing the exact and approximate representations (14) and (15) of the temperature function, we have the inequality

$$\left| T(r, \varphi) - \tilde{T}(r, \varphi) \right|$$

$$\le T_0 \left[\sum_{-n}^{n} \left| D_k(r, \varphi) \right| \left| u_k(\varphi_k) - \tilde{u}_k \right| + \sum_{-n}^{n} C_k \left| u_k(\varphi_k) - \tilde{u}_k \right| + \left| \overset{*}{E}_u(\varphi) \right| + \left| E_c \right| + \left| \overset{*}{E}_f(\varphi) \right| \right]. \tag{17}$$

Successively we obtain

$$\sum_{-n}^{n} \left| D_k(r, \varphi) \right| = \sum_{-n}^{n} D_k(r, \varphi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \ln \frac{2}{\left| t - \exp(i\varphi) \right|} d\tau = 2 \ln 2 ;$$

$$\left| \overset{*}{E}_u(\varphi) \right| - \left| \frac{1}{\pi} \int_{-\pi}^{\pi} u(\tau) \ln \frac{2}{\left| t - \exp(i\varphi) \right|} d\tau - \sum_{-n}^{n} D_k(r, \varphi)\, u_k(\varphi_k) \right| \le 2 \ln 2\omega(u, h) ;$$

$$\left| \overset{*}{E}_f(\varphi) \right| - \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \ln \frac{2}{\left| t - \exp(i\varphi) \right|} d\tau - \sum_{-n}^{n} D_k(r, \varphi)\, f_k(\varphi_k) \right| \le 2 \ln 2\omega(f, h) .$$

It is noted above that

$$\sum_{-n}^{n} \left| C_k \right| = \sum_{-n}^{n} C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\tau}{q(\tau)} = I , \quad \left| E_c \right| \le I\omega(u, h) .$$

Using the last relations and inequality (13) we obtain the estimate of the error (16) on the basis of inequality (17).

N o t e. Inequality (16) yields that when the condition

$$(2 \ln 2 + I)\, q < 1$$

TABLE 1. Exact and Approximate Values of Temperature at T_0 = 1

r	Formula (18)	Formula (15)		
		n = 10	n = 20	n = 50
0.1	0.002426	0.002360	0.002414	0.002423
0.2	0.009705	0.009442	0.009656	0.009692
0.3	0.021832	0.021241	0.021722	0.021807
0.4	0.038793	0.037745	0.038597	0.038742
0.5	0.060549	0.058920	0.060244	0.060469
0.6	0.087019	0.08470	0.086585	0.086905
0.7	0.118060	0.114963	0.117481	0.117908
0.8	0.153440	0.149528	0.152706	0.153247
0.9	0.192815	0.188253	0.191923	0.192581

is observed, the convergence of the numerical algorithm will be ensured if the coefficients in the boundary condition are continuous.

Example. As an example we use the following boundary-value problem:

$$\Delta T = 0\,, \quad r < 1\,,$$

$$\lambda \left.\frac{\partial T}{\partial r}\right|_{r=1} = \alpha \frac{2\sqrt{2}}{3 + \cos 2\varphi} \left[T(r, \varphi)\,|_{r=1} - \frac{-\sin 2\varphi}{\sqrt{2}\,(3 + \cos 2\varphi)} T_{\mathrm{m}} \right].$$

For the sake of definiteness, we set the coefficient Bi = 1. We have obtained the exact expression for the temperature distribution $T = T(r, \varphi)$ in the unit circle

$$T(r, \varphi) = T_0 \frac{\sqrt{2}\,(3 + \sqrt{8})\, r^2 \sin 2\varphi}{(r^2 \cos 2\varphi + 3 + \sqrt{8})^2 + r^4 \sin^2 2\varphi}\,. \tag{18}$$

The results of computations from the exact formula (18) and using the approximate formula (15) for $\varphi = \pi/4$ and n = 10, 25, and 50 are given in Table 1.

The numerical experiment confirms the efficiency of the proposed approximate method of solution of the third boundary-value problem for a circle.

Conclusions. We have established the relationship of the third boundary-value problem of heat-conduction theory for a circle and the integral equation with a logarithmic kernel of a special form. Using the stable computational scheme of solution of such an equation and the special quadrature formulas for integrals with a logarithmic singularity we have obtained the efficient approximate representation of the solution of the third boundary-value problem of heat-conduction theory for a circle with a variable heat-transfer coefficient.

NOTATION

A_k, C_k, and D_k, coefficients of the quadrature formulas; a, b, f, q, and u, functions; Bi, Biot similarity number; C and I, constants; E and $\tilde{E}$, remainders of the quadrature formulas for solution of the integral equation and the boundary-value problem respectively; h, integration step; L^2 and N^2, values of the imaginary parts of the Euler dilogarithm; m, number of values of the functions; n, number of equations in the system; r, polar coordinate; s, order of the polylogarithm; T, temperature, K; t and z, complex variables; α, heat-transfer (heat-exchange) coefficient, W/(m^2·K); Δ, Laplace operator; θ, dimensionless (relative) temperature; λ, thermal conductivity, W/(m·K); φ, angular coordinate; τ, integration variable; ω, moduli of continuity of the functions. Subscripts: m, medium; c, f, and u, indices of the remainders; k, summation index; m, index of values of the functions; 0, characteristic (temperature); $\tilde{}$, approximate value.

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